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The Upper Critical Dimension of the Abelian Sandpile Model

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The existing estimation of the upper critical dimension of the Abelian Sandpile Model is based on a qualitative consideration of avalanches as self-avoiding branching processes. We find an exact representation of an avalanche as a sequence of spanning subtrees of two-component spanning trees. Using equivalence between chemical paths on the spanning tree and loop-erased random walks, we reduce the problem to determination of the fractal dimension of spanning subtrees. Then the upper critical dimension $d_u = 4$ follows from Lawler's theorems for intersection probabilities of random walks and loop-erased random walks.

KEY WORDS: Self-organized criticality; sandpiles; spanning trees; intersection probabilities; upper critical dimension.

1. INTRODUCTION

The standard sandpile model introduced in ref. 1 became of mathematical interest after the paper[2] by Dhar who discovered its Abelian structure. The model is defined on a finite hypercubic d-dimensional lattice. Each site i is characterized by a nonnegative integer variable z_i called the "height." If $z_i \geqslant 2d$ for all i, the sandpile is said to be stable. A vertex is picked at random and its height is increased by one. If $z_i \geqslant 2d$, then the site is unstable and topples giving one particle to each of its neighbols which in their turn, can be unstable and topple. The process called "avalanche" continues until all sites become stable and then a new particle is added to the lattice. If the avalanche reaches the boundary, sand disappears in a sink connected with all boundary sites.

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The "size" of an avalanche may be measured by the total number of topplings s, the number of distinct sites toppled a, the diameter of the region affected by avalanche r, and duration of avalanche t. It is generally believed, the probability that the avalanche has size x ($x = s, a, r, t$) varies asymptotically as $x^{-\tau}$ ($\tau = \tau_s, \tau_a, \tau_r, \tau_t$) when $x \to \infty$. The mean-field value of the exponent τ_s obtained from exact solutions on the Bethe lattice[3] and on the full graph[4] is 3/2. It is expected that $\tau_s = 3/2$ also for $\mathbf{Z}^d$ when $d > d_u$ where d_u is the upper critical dimension.

The first attempt to find d_u was made by Obukhov[5] soon after the sandpile model was proposed. He argued that avalanches are like branching True Self-Avoiding Walks (TSAW). Using the renormalization group and ε-expansion, Obukhov[5] calculated one-loop corrections to the mean-field theory and came to the value $d_u = 4$. This conclusion was supported by Dias-Guilera[6] who analyzed non-linear stochastic differential equations derived from the models with continuously distributed heights.[7] Later on, Christensen and Olami[8] suggested $d_u = 6$ from an analogy between spreading of avalanches and percolation. The mean-field treatment of a self-organized branching process was discussed in ref. 9. Very recently, Vespignani *et al.*[18] have derived $d_u = 4$ from a phenomenological field theory, reflecting the symmetries and conservation laws of sandpiles.

The arguments in favour of correspondence between the branching TSAW and an avalanche process in the sandpile model have two considerable defects. First, a typical avalanche contains multiple topplings which violate self-avoidance of branches. In higher dimensions, multiple topplings are suppressed.[16, 15] However, then, the second problem arises. The branching process corresponding to an avalanche depends deterministically on a recurrent configuration where it was initiated. All recurrent states in the Abelian sandpile model have equal probabilities.[9] If self-avoiding branching processes having equal number of steps, have equal statistical weights, they can be considered as lattice trees also known as branched polymers. The upper critical dimension of branched polymers $d_u = 8$ was predicted in ref. 10 and then rigorously determined in refs. 11–13.

Thus, avalanches either do not correspond to the branching TSAW due to multiple topplings, or do not have the specific TSAW weights due to equal probabilities of recurrent states.

The computation situation is not less controversial. For the most part of numerical experiments, the accuracy is not sufficient to distinguish between non-trivial and mean-field values of critical exponents in high dimensions. Only high-statistics data on large lattices provide some information. Grassberger and Manna[14] investigated critical exponents for $d \leqslant 5$ and concluded that $d_u = 4$. Lübeck and Usadel[15] found $d_u = 4$, mentioning that the largest considered size of the system ($L = 80$) for $d = 4$ is too small

and the corresponding avalanche distribution exhibits a very narrow power-law interval. Chessa *et al.*[17] performed a numerical study of critical exponents in dimensionality ranging from $d=2$ to $d=6$ and observed the mean-field behavior starting from $d=6$.

In his paper,[16] Lübeck pointed an important role of logarithmic corrections to the scaling at $d_u=4$ and, thus, explained why the standard analysis[17] overestimates the upper critical dimension.

An apparent inconsistency between the simple self-avoiding branching process and real avalanches, as well as contradictory numerical results of different groups, set to find a more transparent proof of the upper critical dimension for sandpiles.

In this paper, we prove the upper critical dimension $d_u=4$ using Lawler's theorems[19] for intersection probabilities of random walks and loop-erased random walks. We introduce again the self-avoiding branched polymers for description of avalanches and show that avalanches are spanning subtrees embedded into a spanning tree of the whole lattice rather than usual lattice trees. The problem of fractal dimension of avalanches is reformulated as that for the spanning subtrees. Using Majumdar's result[20] of the equivalence between the chemical path on a spanning tree and the loop-erased random walk,[21] we reduce the problem to estimations of intersection probability between random walks and loop-erased random walks.

2. THE MODEL

We consider the sandpile model on the d-dimensional hypercube $\Lambda \subset \mathbf{Z}^d$. Elements of the state space $\{0, 1, 2,..., (2d-1)\}^{\Lambda}$ are called stable configurations and are denoted by C. The value $C(i)=z_i$, $i \in \Lambda$ is the height of the sandpile at the site i. Given a configuration C and a lattice site i, $a_i C$ is the stable configuration obtained by adding a particle at i, and relaxing the system by topplings at all unstable sites j, $z_j \geqslant 2d$. On toppling at the site j,

$$z_i \to z_i - \Delta_{ij} \qquad i \in \Lambda \tag{1}$$

Elements of the matrix Δ are: $\Delta_{ii}=2d$ for all $i \in \Lambda$; $\Delta_{ij}=-1$ for all bonds (i, j), $|i-j|=1$.

The operators a_i commute with each other[2]

$$[a_i, a_j]=0 \qquad i, j \in \Lambda \tag{2}$$

This allows one to define the identity operator[2]

$$I_i = \prod_{j \in \Lambda} a_j^{\Delta_{ij}} \qquad i \in \Lambda \tag{3}$$

and the equivalence relation between two configurations C' and C''

$$C'(j) = C''(j) + \sum_i n_i \Delta_{ij} \qquad j \in \Lambda \tag{4}$$

where $n_i, i \in \Lambda$ are integers. The equivalence means that both C' and C tend to the same stable configuration after topplings at all unstable sites.

The set of stable configurations $\{C\}$ splits into two subsets: $\{C\} = \{C\}_R \cup \{C\}_T$ where $\{C\}_R$ is the recurrent set and $\{C\}_T$ is the transient set of the sandpile process. In $\{C\}_R$, one call define an inverse operator a_i^{-1}, $i \in \Lambda$. Then, the operators $\{a_i\}$ form a finite Abelian group.[2]

Due to Eq. (4), each $C \in \{C\}_R$ is an element of the super-lattice in the $|\Lambda|$-dimensional Euclidean space. The basis-vectors of this lattice are the rows of the matrix Δ. Therefore, the number of elements in $\{C\}_R$ is[2]

$$N_R = \mathrm{Det}\, \Delta \tag{5}$$

The recurrent configurations in $\{C\}_R$ have equal probabilities N_R^{-1}.

Denote by $\partial\Lambda$ the set of boundary sites of Λ. Let B be the set of unit cubes centered at $i \in \partial\Lambda$ which are elementary cells of the d-dimensional hypercubic lattice. We choose a site $\hat{\imath} \in \mathbf{Z}^d$ not belonging to Λ and call it the root. The faces of cubes from B which can be connected with i by a continuous line without intersections with another face, are called external faces. We connect each site $i \in \partial\Lambda$ with $\hat{\imath}$ by v_i bonds where v_i is the number of external faces of the cube i. The lattice Λ together with $\hat{\imath}$ and new bonds forms a graph denoted by G. The site $\hat{\imath}$ is a sink for particles leaving the lattice Λ during the avalanche.

Using the explicit form of the identity operator Eq. (3), one can construct a new identity operator

$$I_{\partial\Lambda} = \prod_{i \in \partial\Lambda} a_i^{v_i} \tag{6}$$

$I_{\partial\Lambda}$ says that adding v_i particles to each boundary site triggers an avalanche of a special form: each site $i \in \Lambda$ topples exactly once and the initial configuration C remains unchanged. One can use $I_{\partial\Lambda}$ to construct a graph representation for any $C \in \{C\}_R$ by the so-called "burning algorithm".[22] Consider the topplings initiated by $I_{\partial\Lambda}$ as a fire starting at $\hat{\imath}$ and burning sequentially all lattice sites. Once the rules of propagation of fire are fixed,

the set of bonds along which fire propagates forms a subgraph T of the graph G with the root at $\hat{\imath}$.

By definition, a connected subgraph T is the rooted one-component spanning tree of the graph G if the sites of T coincide with those of G, the subgraph T has no loops and there is a selected site of T called the root.

There is one-to-one correspondence between $\{C\}_R$ and the set of spanning trees T. Then, Dhar's formula, Eq. (5), for N_R coincides with Kirhhoff's theorem for the number of spanning trees.[23]

In addition to the one-component spanning trees, we introduce subgraphs containing more than one connected component. Given the set of v vertices $\{v\} = i_1, i_2, ..., i_v$ we call the subgraph $T_{\{v\}}$ the v-component rooted spanning tree if $T_{\{v\}}$ consists of v connected components which are mutually disconnected, has no loops and has on of v vertices $i_1, i_2, ..., i_v$, called the roots in each component.

The matrix Δ in Eq. (5) is the minor of the Laplacian matrix Δ_G of the graph G corresponding to the element $(\Delta_G)_{\hat{\imath}\hat{\imath}}$. According to Dhar,[2] the Green function

$$G_{ij} = [\Delta^{-1}]_{ij} \tag{7}$$

is the average number of topplings at the site j due to a single particle added at i in a configuration $C \in \{C\}_R$. For the random walk defined by the matrix Δ, G_{ij} is the expected number of times the walk started at i visits the site j until it is trapped by the absorbing site $\hat{\imath}$.[26]

To find the spanning tree representation for G_{ij}, we delete from Λ the unit cube centered at i and consider its $2d$ faces as a part of the new external boundary. The sites adjacent to i form a new boundary set a $\partial^i\Lambda$. Repeating the construction of a spanning tree by the identity operator, Eq. (6), with the boundary set $\partial\Lambda \cup \partial^i\Lambda$, we obtain a two-component spanning tree having the roots $\hat{\imath}$ and i for two different subtrees.

Proposition 1 (Proved in ref. 24; see also ref. 25). For any connected graph G with a fixed vertex $\hat{\imath}$

$$G_{ij} = N^{(i,j)}/|T| \tag{8}$$

where $N^{(i,j)}$ is the number of two-component spanning trees having the roots $\hat{\imath}$ and i, such that both vertices i and j belong to the same component; $|T|$ is the number of one-component spanning trees on G.

3. AVALANCHES AND WAVES

An avalanche starting at i is the process of transformation $C \to a_i C$. Generally, an avalanche consists of multiple topplings at different sites and

has a complicated structure. However, one can try to decompose it into simpler subprocesses. Suppose $z_i = 2d$ after adding one particle to i. Since the topplings can be performed in any order, we topple once at i, and then topple all other unstable sites keeping the site i out of the next toppling. This sub-avalanche is the first wave of topplings. If i is still unstable, we topple i once again giving rise to the second wave of topplings. This process is continued until i becomes stable. Thus, an avalanche is broken into a sequence of waves of topplings.

Three important properties of waves make them useful for the analysis of avalanche statistics.[24] Let S_k, $k \geqslant 1$ be a set of sites toppled during the kth wave. Then,

(i) each $j \in S_k$ topples once and only once;

(ii) both S_k and $\Lambda \backslash S_k$ are connected sets;

(iii) kth wave is the last wave of an avalanche iff the initial site $i \in S_k$ has a neighbor in $\Lambda \backslash S_k$.

The number of waves in the avalanche $C \to a_i C$ will be denoted by $n_i(C)$ or simply n_i.

The spanning tree representation of waves can be constructed as follows. Given a configuration $C \in \{C\}_R$, $P_i(C)$ represents the projection of C on the lattice $\Lambda \backslash i$ where the site i is now considered as the second root. Let C_k, $k \leqslant n_i$ be an (unstable) configuration obtained after the kth wave starting at i is completed. Then, $P_i(C_k)$ can be produced from $P_i(C)$ by the operator

$$W_k = \left(\prod_{j \in \partial^i \Lambda} a_j \right)^k \tag{9}$$

acting on the lattice $\Lambda \backslash i$.

Consider the configuration

$$P_i(C_{k-1}) = W_{k-1} P_i(C) \tag{10}$$

preceeding the kth wave. If one applies to $P_i(C_{k-1})$ the identity operator

$$I_{\partial \Lambda \cup \partial^i \Lambda} = \left(\prod_{j \in \partial^i \Lambda} a_j \right) \left(\prod_{j \in \partial \Lambda} a_j^{v_j} \right) \tag{11}$$

one obtains a two-component spanning tree representing $P_i(C_{k-1})$. On the other hand, the expression in the first brackets is exactly the operator providing the kth wave. Therefore, the subtree rooted at i and embedded into the subtree rooted at $\hat{\imath}$, is the graph representation of the kth wave.

This result gives another proof of Eq. (8). As each wave beginning at i and involving j corresponds to one toppling at j, $N^{(i,j)}/|T|$ is the average number of topplings at the site j for a single particle added at i. Then, Eq. (8) follows from Eq. (7).

Due to the graph representation, we come again to a lattice-tree or a branched-polymer picture of avalanches. However, the essential difference with the qualitative Obuhkov arguments is, firstly, that we associate the lattice trees with waves but not with the whole avalanches, and secondly, the lattice trees are conditioned by the spanning tree construction.

Let s_k denote the number of sites in the kth wave S_k. The total number of topplings s in an avalanche is

$$s = \sum_{k=1}^{n_i} s_k \tag{12}$$

where n is the number of topplings at the initial point i or, equivalently, the number of waves in the avalanche. It is expected that, at least for sufficiently large dimensions d, the probability distribution of s has the scaling form

$$P(s; L) \sim s^{-\tau} f(s/L^D) \tag{13}$$

where D is the capacity fractal dimension of avalanches. It follows from Eq. (12) that D depends on the fractal dimension of waves and the number of waves in an avalanche. In $d=2$, the dimension of waves is $d_w = 2$ and the number of waves is expected to obey the scaling law[22]

$$\langle n_i \rangle \sim r^y \tag{14}$$

Then the exponents τ_s, τ_a, τ_r, τ_t can be determined in terms of y. In higher dimensions, multiple toppling events occurs more rarely showing a tendency to decrease y. In Section 7, we will show that for $d=4$, $\langle n_i \rangle$ grows with r not faster than logarithmically. If so, the problem of critical dimension for avalanches can be formulated as that for waves.

4. STATISTICS OF WAVES

We start this section with the definition of the fractal dimension of waves. Consider the graph representation of a wave as a two-component spanning tree on the graph G with the roots $i \in \Lambda$ and $\hat{\imath}$. The wave S is the

set of $s = |S|$ sites belonging to the component rooted at i. The radius of the wave is

$$R(S) = \sup\{d(i, j): j \in S\} \tag{15}$$

where $d(i, j)$ is the distance between the sites i and j. We will say that waves have a fractal growth if

$$\langle s \rangle \sim R^{d_{fg}} \tag{16}$$

for large R, where d_{fg} is the dimension of fractal growth and the average is taken over all waves of the radius R. The fractal growth, however, does not guarantee the fractal structure by itself.

Surround the point i by a ball of radius r and consider all sites of S inside the ball. Denote the number of internal points by $m(r)$. We say that waves have the fractal density if, given $\varepsilon > 0$, there exist $r(\varepsilon)$, $R(\varepsilon)$ and a constant $\bar{c} < 1$ such that

$$\text{const} - \varepsilon \leqslant \log \langle m(r) \rangle - d_{fd} \log r \leqslant \text{const} + \varepsilon \tag{17}$$

on the interval $r(\varepsilon) \leqslant r < \bar{c}R$ for all $R > R(\varepsilon)$.

If $d_{fg} = d_{fd} = d_f$, the fractal is uniform and d_f is the fractal dimension of waves on the d-dimensional lattice. We do not prove here uniformity of the fractal structure of waves. Assuming it, we will find the upper and lower bounds for d_f.

In the spherically symmetrical case, we can introduce the density of waves which varies as

$$\rho(r) \sim \frac{1}{r^{d-d_f}} \tag{18}$$

in the scaling interval $r(\varepsilon) \leqslant r \leqslant \bar{c}R$.

If d_f is known, one can determine the probability distribution of waves $P(x)$, $x = s, a, R, t$ ($s = a$ for waves). Indeed, the expected number of topplings at a point j in an avalanche starting at i, $|i - j| = r$ varies as

$$\langle n(r) \rangle \sim \rho(r) \int_{r/\bar{c}}^{\infty} P(R)\, dR \tag{19}$$

where the integral gives all waves having reached the radius $r/\bar{c}$. On the other hand, $\langle n(r) \rangle$ coincides with the Green function, Eq. (7). The asymptotics of $G(r)$ in the d-dimensional case is

$$G(r) \sim \frac{1}{r^{d-2}} \tag{20}$$

Comparing with Eq. (19), we find that

$$P(R) \sim \frac{1}{R^{d_f - 1}} \tag{21}$$

The probability distribution for s follows from Eq. (16) provided that $s \sim R^{d_f}$ when $R \to \infty$

$$P(s) \sim \frac{1}{s^{\tau_s}} \tag{22}$$

with $\tau_s = 2 - 2/d_f$. The exponent τ_s reaches its mean-field value 3/2 when $d_f = 4$.

To proceed with the determination of d_f, we need a more elaborate definition of $\rho(r)$. Consider a wave S with the initial site i. The set of sites not belonging to S is denoted by $\hat{S}$, $S \cup \hat{S} = \Lambda$. According to Eq. (15), a wave has the radius not less than R if there exists a site $i' \in S$, $|i - i'| = R$. The wave S has the density less than 1 at the radius $r \leqslant R$ if there exists a site $\hat{i}' \in \hat{S}$, $|i - \hat{i}'| = r$. Consider the two-component spanning tree with the components corresponding to S and $\hat{S}$. The path $\Gamma(ij)$ on a tree between the points i and j is the sequence of bonds (i, i_1), (i_1, i_2),..., (i_n, j). Any two points belonging to one component can be connected by a unique path. Then, we can define the density of waves at the radius $r \leqslant R$ by the conditional probability that, given a path $\Gamma(ii')$, $|i - i'| = R$, there exists a path $\Gamma(\hat{i}\hat{i})$, $|i - \hat{i}'| = r$,

$$\rho(r) = 1 - \text{Prob}(\Gamma(\hat{i}\hat{i}') \mid \Gamma(ii')) \tag{23}$$

The expression in the right-hand side of Eq. (23) is the non-intersection probability of two self-avoiding paths on a tree, called often "chemical paths." The first path $\Gamma(ii')$ connects the site i with the site i' belonging to the wave S; the second path $\Gamma(\hat{i}\hat{i}')$ connects the site $\hat{i}'$ not belonging to S with the sink $\hat{i}$. Having no rigorous results for this probability, we can, however, reduce it to non-intersection probability between one chemical path and the simple random walk.

For this purpose, we need a generalization of Proposition 1 to the multicomponent case. Consider a connected graph G and select a subset of its vertices $\{v\} = i_1, i_2, ..., i_v$. Let $\Delta_{\{v\}}$ be the matrix obtained from the Laplacian matrix Δ_G by deleting all rows and columns corresponding to $i_1, i_2, ..., i_v$. The Green function

$$G_{ij} = [\Delta_{\{v\}}^{-1}]_{ij} \tag{24}$$

is the expected number of visits of the site j by the random walk starting at i before it is absorbed at one of the sites $i_1, i_2, ..., i_v$.

Proposition 2. For any connected graph G with a fixed subset of vertices $\{v\}$

$$G_{ij} = \frac{N^{(ij)}_{\{v\}}}{|T_{\{v\}}|} \tag{25}$$

where $N^{(ij)}_{\{v\}}$ is the number of $(v+1)$-component spanning trees having the roots $i_1, i_2, ..., i_v$ and i such that vertices i and j belong to the same component; $|T_{\{v\}}|$ is the number of v-component spanning trees on G having the roots $i_1, i_2, ..., i_v$.

Remark. If a site $i_\mu \in \{v\}$ is an isolated site of the spanning tree, we consider it as a component consisting of the single site.

Proof. Consider the graph G' which is the graph G with all sites $i_1, i_2, ..., i_v$ contracted to the single site i_0. Proposition 1 is applicable to G'. It relates the Green function, Eq. (24), with the number of two-component spanning trees having the roots i_0 and i, such that the sites i and j belong to the same component. Returning to the graph G, we get from each configuration on G', a spanning tree where the component containing i_0 splits into v components. This construction implies Proposition 2.

In order to treat the right-hand side of Eq. (23) as a random walk probability, we consider the graph G with the subset of vertices $\{v\}$ coinciding with the set of all sites of the chemical path $\Gamma(ii')$. Then, by Proposition 2, we have

$$G_{\hat{i}'\hat{i}} = \frac{N^{(\hat{i}\hat{i}')}_{\{v\}}}{|T_{\{v\}}|} \tag{26}$$

where $G_{\hat{i}'\hat{i}}$ is the expected number of visits of the site $\hat{i}$ by the random walk starting at $\hat{i}'$ and escaping the chemical path $\Gamma(ii')$. If v is the number of sites in $\Gamma(ii')$ then $N^{(\hat{i}\hat{i}')}_{\{v\}}$ is the number of $(v+1)$ component spanning trees having sites $\hat{i}$ and $\hat{i}'$ in one component. The remaining v components can be joint into a single component if one adds to v spanning subtrees rooted at the sites of $\Gamma(ii')$, the path $\Gamma(ii')$ itself. Then, $N^{(\hat{i}\hat{i}')}_{\{v\}}$ is the number of two-component spanning trees which have the path $\Gamma(ii')$ in one component corresponding to the wave S and the path $\Gamma(\hat{i}\hat{i}')$ in the second component $\hat{S}$ having the root $\hat{i}'$ or, equivalently, $\hat{i}$. $|T_{\{v\}}|$ is the number of all one-component spanning trees containing the path $\Gamma(ii')$.

Along with $|T_{\{v\}}|$, we introduce $N^{(i)}_{\{v\}}$, the number of two-component spanning trees containing the path $\Gamma(ii')$ in one component and the root $\hat{\imath}$ in the second one. By definition,

$$\mathrm{Prob}(\Gamma(\hat{\imath}\hat{\imath}') \mid \Gamma(ii')) = \frac{N^{(\hat{\imath}\hat{\imath}')}_{\{v\}}}{N^{(\hat{\imath})}_{\{v\}}} = \frac{N^{(\hat{\imath}\hat{\imath}')}_{\{v\}}}{|T_{\{v\}}|} \frac{|T_{\{v\}}|}{N^{(\hat{\imath})}_{\{v\}}} \tag{27}$$

Using Eq. (26), we get

$$\mathrm{Prob}(\Gamma(\hat{\imath}\hat{\imath}') \mid \Gamma(ii')) = \frac{G_{\hat{\imath}'\hat{\imath}}}{G_{\hat{\imath}\hat{\imath}}} \tag{28}$$

where the Green function $G_{\hat{\imath}\hat{\imath}}$ is the expected number of returns to the initial point for random walks starting at $\hat{\imath}$ and escaping the path $\Gamma(ii')$.

Finally we note that the ratio of Green functions in Eq. (28) is the probability that the random walk starting at $\hat{\imath}'$ escapes the path $\Gamma(ii')$ and reaches the site $\hat{\imath}$. This fact follows from the property of Green functions known for infinite lattices as the "weak ergodic theorem".[26] Denoting by $F(\hat{\imath}' \mid ii')$ the escaping probability averaged over all paths $\Gamma(ii')$, we can summarize the obtained results as

Proposition 3. The waves starting at a site i of the lattice Λ with the sink $\hat{\imath}$ have the density

$$\rho(r) = 1 - F(\hat{\imath}' \mid ii') \tag{29}$$

at the point $\hat{\imath}'$, $|i - \hat{\imath}i| = r$, if the radius of the wave exceeds $R = |i - i'| \geqslant r$.

5. LOOP ERASED RANDOM WALKS

The loop erased random walk (LERW) has been introduced by Lawler[21] to modify the uniform measure of the usual self-avoiding random walk where every possible walk of a fixed length is given the same statistical weight. Let $[x_0, ..., x_m]$ be the simple random walk in $\mathbf{Z}^d$. Then the LERW can be constructed as follows: let j be the smallest value such that $x_j = x_i$ for $0 \leqslant i < j \leqslant m$. Deleting all steps between i and j, we obtain a new walk $[x_0, ..., x_i, x_{i+1}, ..., x_m]$. If the new walk is self-avoiding we stop the process; otherwise, we continue the loop-erasing operation until we get a self-avoiding path.

The distribution for each step of the LERW depends on the entire past history being equivalent to that given by the transition probability of the Laplacian random walk.[27, 28] Another equivalence has been found by

Majumdar[20] who considered a model of growing trees introduced, by Broder.[29] The chemical path on these trees corresponds to the LERW. Like the burning algorithm, Broder's algorithm in the limit of lattice filling gives spanning trees with equal probability. Therefore, the statistical properties of the LERW are identical to those of the chemical path on spanning trees.

Let $R_n(i)$ and $L_n(i)$ be simple and loop-erased random walks starting at the site i and let Π_i and Γ_i denote paths of the walks

$$\Pi_i(a) = \{R_n(i): 0 \leqslant n \leqslant a\} \tag{30}$$

$$\Gamma_i(a) = \{L_n(i): 0 \leqslant n \leqslant a\} \tag{31}$$

Denote by $\sigma(n)$ the number of steps of the simple random walk needed to produce n steps of the LERW

$$\sigma(n) = \sup\{j: L_n(i) = R_j(i)\} \tag{32}$$

For any random walk, $\sigma(n) \geqslant n$. For a fixed $\sigma(n)$, $\Pi_i(\sigma(n))$ is the path of a finite random walk. Then, $\Gamma_i(n)$ is the finite LERW obtained from $\Pi_i(\sigma(n))$ by the loop-erasing procedure.

The escaping probability of a set $A \in \mathbf{Z}^d$ is defined as

$$Es(i, A) = \text{Prob}\{R_n(i) \notin A, n = 1, 2, 3, ...\} \tag{33}$$

where i is the starting point of the random walk. Putting $A = \Gamma_i(n)$, we can define a_n as

$$a_n = E[Es(i, \Gamma_i(n))] \tag{34}$$

i.e., a_n is the probability that an infinite independent simple random walk starting in $i \in \mathbf{Z}^d$ does not intersect the finite LERW derived by erasing loops from $\sigma(n)$ steps of an independent random walk also starting in i.

For $d = 4$, Lawler[30] has found the lower and upper bounds for a_n

$$\lim_{n \to \infty} \inf \frac{\log a_n}{\log \log n} \geqslant -\frac{1}{2} \tag{35}$$

and

$$\lim_{n \to \infty} \sup \frac{\log a_n}{\log \log n} \geqslant -\frac{1}{3} \tag{36}$$

The upper bound Eq. (36) was conjectured[30] to coincide with the exact asymptotics. This result would imply that the mean square distance for LERW's behaves as

$$\langle r_n^2 \rangle \sim n(\log n)^{1/3} \tag{37}$$

whereas $\langle r_n^2 \rangle \sim n$ for $d \geqslant 5$.

It is convenient to bring here the known intersection probability for two simple random walks.[19] Let $R_n(i)$ and $R_n(j)$ be independent random walks starting at the points i and j, $|i-j|=r$. Then, there exist constants $c_1>0$ and $c_2>0$ such that

$$c_1 \leqslant \mathrm{Prob}\{\Pi_i(n) \cap \Pi_j(n) \neq 0\} \leqslant c_2 \tag{38}$$

for $d<4$

$$c_1(\log n)^{-1} \leqslant \mathrm{Prob}\{\Pi_i(n) \cap \Pi_j(n) \neq 0\} \leqslant c_2(\log n)^{-1} \tag{39}$$

for $d=4$

$$c_1 n^{(4-d)/2} \leqslant \mathrm{Prob}\{\Pi_i(n) \cap \Pi_j(n) \neq 0\} \leqslant c_2 n^{(4-d)/2} \tag{40}$$

for $d>4$, if $a\sqrt{n} \leqslant |r| \leqslant b\sqrt{n}$, where a and b are positive constants.

The non-intersection probability of a finite random walk and an infinite random walk starting at the same point i for $d-4$ is[19, 32]

$$\mathrm{Prob}\{\Pi_i(n) \cap \Pi_i(\infty) = 0\} \sim c(\log n)^{-1/2} \tag{41}$$

Also, for further estimations, we need the intersection probabilities between a finite random walk starting at i and an infinite random walk starting at j for $d=4$[31]

$$\mathrm{Prob}\{\Pi_i(n) \cap \Pi_j(\infty) \neq 0\} \leqslant c_1 \log(1+1/\alpha)/\log n \tag{42}$$

if $\alpha = |i-j|^2/n$.

6. UPPER AND LOWER BOUNDS FOR DENSITY OF WAVES

Every random walk on a graph G with an absorbing set $\{v\}$ and the sink $\hat{\imath}$ is trapped either by $\{v\}$ or by $\hat{\imath}$. Then, Proposition 3 means that $\rho(r)$ is the intersection probability between the random walk starting at the site $\hat{\imath}$, $|i-\hat{\imath}'|=r$ and the chemical path $\Gamma(ii')$, $|i-i'|=R$.

Majumdar[20] has proved that chemical paths correspond to LERW's which, in turn, can be obtained from simple random walks by the loop-erasing procedure. Let $R_n(i)$ be the random walk corresponding to the

LERW $L_n(i)$. As the number of steps $\sigma(n)$ in the path $\Pi_i(\sigma(n))$, Eq. (30), needed to reach the radius R, exceeds the number of steps n in the LERW path $\Gamma_i(n)$, Eq. (31), we can estimate $\rho(r)$ as

$$\rho(r) \leqslant \mathrm{Prob}\{\Pi_i(\sigma(n)) \cap \Pi_{\hat{\imath}'}(\infty) \neq 0\} \tag{43}$$

In $d=4$, we get from Eq. (42)

$$\rho(r) \leqslant c \log\left(1+\frac{1}{\alpha}\right) \Big/ \log r \tag{44}$$

where $\alpha = r^2/\sigma(n) \sim r^2/R^2$. The fractal density of waves in $d=4$ decays with r at least logarithmically when α is fixed. In $d \geqslant 4$, we can use Eq. (40) assuming that the asymptotics does not change when one of the random walks is extended to infinity. Then, we have

$$\rho(r) \leqslant c r^{4-d} \tag{45}$$

From the definition Eq. (18), we can see that the fractal dimension of waves $d_f \leqslant 4$ for all $d \geqslant 4$.

In $d \leqslant 4$, the intersection probability, Eq. (38), and, therefore $\rho(r)$, are restricted from above by a constant.

To get a lower bound for $\rho(r)$ in $d=4$, we will use the upper bound Eq. (36) for escaping probability a_n. Consider the random walk $R_n(i)$ and decompose its path $\Pi_i(\infty)$ into two parts: $\Pi_i(\infty) = \Pi_i(k) \cup \Pi_{\hat{\imath}'}(\infty)$ where k is the moment of the first hitting into the site $\hat{\imath}'$ separated from i by the distance r: $R_i(k) = \hat{\imath}'$, $R_i(l) \neq \hat{\imath}'$, $l = 1, 2, \ldots, k-1$. Then, escaping probability a_n for the LERW of length n by the random walk $R_n(i)$ is the product of escaping probabilities $a_n(i, \hat{\imath}')$ and $a_n(\hat{\imath}', \infty)$ before and after the first hitting into the site $\hat{\imath}'$

$$a_n = a_n(i, \hat{\imath}')\, a_n(\hat{\imath}', \infty) \tag{46}$$

Granting that $a_n(\hat{\imath}', \infty) = F(\hat{\imath}' \mid ii')$, we can write Eq. (29) as

$$\rho(r) = 1 - \frac{a_n}{a_n(i, \hat{\imath}')} \tag{47}$$

where n is the length of the LERX coinciding with the path $\Gamma(ii')$.

The probability $a_n(i, \hat{\imath}')$ to reach the point $\hat{\imath}'$ from i for m steps, $m \sim |i - \hat{\imath}'|^2 = r^2$, escaping the LERW of length n is not less than the probability to avoid the random walk of length $\sigma(n)$ from which the LERW was

obtained by the loop-erasing procedure, and therefore not less than the probability $\mathrm{Prob}\{\Pi_{(}\infty) \cap \Pi_i(m) = 0\}$ to avoid an infinite random walk. Using Eqs. (36) and (41) we get from Eq. (47)

$$\rho(r) \geqslant 1 - c \frac{(\log m)^{1/2}}{(\log n)^{1/3}} \tag{48}$$

For any $r \sim m^{1/2}$, we see from Eq. (37) that the number of steps n in the LERW increases with the radius of wave as R^2 (with the logarithmic correction) and $\rho(r)$ approaches 1 when $R \to \infty$. The only fractal dimension which is consistent with both upper and lower bounds Eq. (44) and Eq. (48) is 4 although the fractal density Eq. (18) as well as the probability distribution of waves Eqs. (21), (22) need logarithmic corrections.

The lower bound Eq. (48) becomes stronger in $d < 4$ because of decreasing escaping probability $F(\hat{\imath}'\, ii')$. Assuming that d_f is a non-decreasing function of d, we conclude that the fractal dimension of waves is

$$d_f = \begin{cases} d & \text{for} \quad d \leqslant 4 \\ 4 & \text{for} \quad d > 4 \end{cases} \tag{49}$$

According to Eq. (22), the upper critical dimension for waves is $d_u = 4$.

7. NUMBER OF WAVES IN AVALANCHES

In order to derive the upper critical dimension for avalanches from that for waves we need some information about the expected number of waves in an avalanche. Due to the property (iii) of waves mentioned in Section 3, the probability that a wave starting at the site i is the last wave in an avalanche is proportional to the density of "holes" in the wave at its origin, i.e., the probability that a site in the vicinity of i does not belong to the wave. By Eq. (29) this probability is $F(\hat{\imath}' \mid i\hat{\imath}')$ which coincides simply with the escaping probability a_n in the case $r = |i - \hat{\imath}'| \to 0$.

Avalanches of radius r consist of the waves whose radius does not exceed r. Therefore, the expected number of waves can be estimated as

$$\langle n_i \rangle \leqslant c a_n^{-1} \tag{50}$$

where $r^2 \sim n(\log n)^{1/3}$. Using Eq. (35) we get

$$\langle n_i \rangle \leqslant c(\log n)^{1/2} \tag{51}$$

or

$$\langle n_i \rangle \leqslant c_1 (\log r)^{1/2} \tag{52}$$

Thus, in $d=4$, the number of waves in an avalanche of radius r grows not faster than logarithmically.

In $d=2$, the distribution of last waves is known[33]

$$P_l(R) \sim \frac{1}{R^{7/4}} \tag{53}$$

The probability for the wave of radius R to be the last in an avalanche is $P_l(R)$ divided by the general distribution of waves Eq. (21). Then, the expected number of waves $\langle n_i \rangle$ in $d=2$ has the upper estimate

$$\langle n_i \rangle \leqslant cr^{3/4} \tag{54}$$

which is consistent with Eq. (14) where the exact value $y=1/2$ was conjectured.[34]

For $d>4$, the escaping probability a_n remains finite for all n. Therefore, the expected number of waves in an avalanche is restricted by a constant.

The logarithmic growth of $\langle n_i \rangle$ in $d=4$ given by Eq. (52) implies that the probability distributions for waves add avalanches can differ not more than by logarithmic correction.

To estimate this correction, we take instead of the density $\rho(r)$ its upper bound Eq. (44) if r.h.s. Eq. (44) is less than 1, and put $\rho(r)=1$ otherwise. Comparing Eq. (20) with Eq. (19) where the new $\rho(r)$ is taken, we get the probability distribution for waves in the form

$$P(R) = \frac{\log R}{R^3} \tag{55}$$

Also, we take instead of the lower bound Eq. (35) for a_n its exact asymptotics conjectured by Lawler.[30] Then,

$$\langle n_i \rangle \sim (\log r)^{1/3} \tag{56}$$

The maximal difference between avalanche and wave distributions corresponds to a situation when all waves in an avalanche of radius r have the maximal radius r

$$\langle n_i \rangle P_{\text{aval}}(r)\, dr = P(r)\, dr \tag{57}$$

Hence, the avalanche distribution in $d=4$ can be estimated as

$$P_{\text{aval}}(r) \sim \frac{(\log r)^{\gamma}}{r^3} \tag{58}$$

with $2/3 \leqslant \gamma \leqslant 1$.

The probability distribution of the total number of topplings in a wave $P(s)$ follows from Eq. (55) provided the leading asymptotic of s is $s \sim R^4/\log R$:

$$P(s) \sim \frac{(\log s)^{1/2}}{s^{3/2}} \tag{59}$$

In $d>4$, the asymptotics of the avalanche and wave distributions coincide and correspond to the mean-field behavior with the exponent $\tau_s = 3/2$.

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